Section A: Pure Mathematics

1 Consider the equations

$$\begin{aligned} ax - y - \ z &= 3\,, \\ 2ax - y - 3z &= 7\,, \\ 3ax - y - 5z &= b\,, \end{aligned}$$

where a and b are given constants.

(i) In the case $a = 0$, show that the equations have a solution if and only if $b = 11$.

(ii) In the case $a \neq 0$ and $b = 11$ show that the equations have a solution with $z = \lambda$ for any given number λ.

(iii) In the case $a = 2$ and $b = 11$ find the solution for which $x^2 + y^2 + z^2$ is least.

(iv) Find a value for a for which there is a solution such that $x > 10^6$ and $y^2 + z^2 < 1$.

2 Write down a value of θ in the interval $\pi/4 < \theta < \pi/2$ that satisfies the equation

$$4\cos\theta + 2\sqrt{3}\,\sin\theta = 5\ .$$

Hence, or otherwise, show that

$$\pi = 3\arccos(5/\sqrt{28}) + 3\arctan(\sqrt{3}/2)\ .$$

Show that

$$\pi = 4\arcsin(7\sqrt{2}/10) - 4\arctan(3/4)\ .$$

3 Prove that the cube root of any irrational number is an irrational number.

Let $u_n = 5^{1/(3^n)}$. Given that $\sqrt[3]{5}$ is an irrational number, prove by induction that u_n is an irrational number for every positive integer n.

Hence, or otherwise, give an example of an infinite sequence of irrational numbers which converges to a given integer m.

[An irrational number is a number that cannot be expressed as the ratio of two integers.]

4 The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H. Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2}\,. \qquad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression $(*)$ should be modified to give the area of the minor segment.

(i) Line: $y = c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

(ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.

(iii) Line: $y = mx + c$; circle: $(x-a)^2 + (y-b)^2 = R^2$.

5 The position vectors of the points A, B and P with respect to an origin O are $a\mathbf{i}$, $b\mathbf{j}$ and $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, respectively, where a, b, and n are all non-zero. The points E, F, G and H are the midpoints of OA, BP, OB and AP, respectively. Show that the lines EF and GH intersect.

Let D be the point with position vector $d\mathbf{k}$, where d is non-zero, and let S be the point of intersection of EF and GH. The point T is such that the mid-point of DT is S. Find the position vector of T and hence find d in terms of n if T lies in the plane OAB.

6 The function f is defined by

$$\mathrm{f}(x) = |x - 1|\,,$$

where the domain is $\mathbf{R}$, the set of all real numbers. The function $\mathrm{g}_n = \mathrm{f}^n$, with domain $\mathbf{R}$, so for example $\mathrm{g}_3(x) = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$. In separate diagrams, sketch graphs of g_1, g_2, g_3 and g_4.

The function h is defined by

$$\mathrm{h}(x) = \left|\sin\frac{\pi x}{2}\right|\,,$$

where the domain is $\mathbf{R}$. Show that if n is even,

$$\int_0^n \big(\mathrm{h}(x) - \mathrm{g}_n(x)\big)\,\mathrm{d}x = \frac{2n}{\pi} - \frac{n}{2}\,.$$

7 Show that, if $n > 0$, then

$$\int_{e^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x = \frac{2}{n^2 e}\,.$$

You may assume that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x < \int_a^{\infty} \frac{\ln x}{x^{n+1}}\,\mathrm{d}x\,.$$

Deduce that

$$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{e}{2}\int_{e^{1/N}}^{\infty} \left(\frac{1 - x^{-N}}{x^2 - x}\right) \ln x\,\mathrm{d}x\,,$$

where N is any integer greater than 1.

8 It is given that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} + k\left(\frac{t^2 - 3t + 2}{t + 1}\right) y = 0\,,$$

where k is a constant, and $y = A$ when $t = 0$, where A is a positive constant. Find y in terms of t, k and A.

Show that y has two stationary values whose ratio is $(3/2)^{6k} e^{-5k/2}$.

Describe the behaviour of y as $t \to +\infty$ for the case where $k > 0$ and for the case where $k < 0$.

In separate diagrams, sketch the graph of y for $t > 0$ for each of these cases.

Section B: Mechanics

9 AB is a uniform rod of weight W. The point C on AB is such that $AC > CB$. The rod is in contact with a rough horizontal floor at A and with a cylinder at C. The cylinder is fixed to the floor with its axis horizontal. The rod makes an angle α with the horizontal and lies in a vertical plane perpendicular to the axis of the cylinder. The coefficient of friction between the rod and the floor is $\tan\lambda_1$ and the coefficient of friction between the rod and the cylinder is $\tan\lambda_2$.

Show that if friction is limiting both at A and at C, and $\alpha \neq \lambda_2 - \lambda_1$, then the frictional force acting on the rod at A has magnitude

$$\frac{W\sin\lambda_1\,\sin(\alpha-\lambda_2)}{\sin(\alpha+\lambda_1-\lambda_2)}\,.$$

10 A bead B of mass m can slide along a rough horizontal wire. A light inextensible string of length 2ℓ has one end attached to a fixed point A of the wire and the other to B. A particle P of mass $3m$ is attached to the mid-point of the string and B is held at a distance ℓ from A. The bead is released from rest.

Let a_1 and a_2 be the magnitudes of the horizontal and vertical components of the intial acceleration of P. Show by considering the motion of P relative to A, or otherwise, that $a_1 = \sqrt{3}a_2$. Show also that the magnitude of the intial acceleration of B is $2a_1$.

Given that the frictional force opposing the motion of B is equal to $(\sqrt{3}/6)R$, where R is the normal reaction between B and the wire, show that the magnitude of the intial acceleration of P is $g/18$.

11 A particle P_1 is projected with speed V at an angle of elevation α $(> 45^\circ)$, from a point in a horizontal plane. Find T_1, the flight time of P_1, in terms of α, V and g. Show that the time after projection at which the direction of motion of P_1 first makes an angle of 45° with the horizontal is $\frac{1}{2}(1-\cot\alpha)T_1$.

A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of 45° with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2, show that

$$\frac{2T_2}{T_1} = 1 + \cot\alpha + \sqrt{1+3\cot^2\alpha}\,.$$

SECTION C: Probability and Statistics

12 The life of a certain species of elementary particles can be described as follows. Each particle has a life time of T seconds, after which it disintegrates into X particles of the same species, where X is a random variable with binomial distribution $\mathrm{B}(2,p)$. A population of these particles starts with the creation of a single such particle at $t=0$. Let X_n be the number of particles in existence in the time interval $nT < t < (n+1)T$, where $n = 1, 2, \ldots$.

Show that $\mathrm{P}(X_1 = 2 \text{ and } X_2 = 2) = 6p^4q^2$, where $q = 1-p$. Find the possible values of p if it is known that $\mathrm{P}(X_1 = 2 | X_2 = 2) = 9/25$.

Explain briefly why $\mathrm{E}(X_n) = 2p\mathrm{E}(X_{n-1})$ and hence determine $\mathrm{E}(X_n)$ in terms of p. Show that for one of the values of p found above $\lim_{n\to\infty} \mathrm{E}(X_n) = 0$ and that for the other $\lim_{n\to\infty} \mathrm{E}(X_n) = +\infty$.

13 The random variable X takes the values $k = 1, 2, 3, \ldots$, and has probability distribution

$$\mathrm{P}(X = k) = A\frac{\lambda^k e^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = (1 - \mathrm{e}^{-\lambda})^{-1}$. Find the mean μ in terms of λ and show that

$$\mathrm{Var}\,(X) = \mu(1 - \mu + \lambda).$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

14 The probability of throwing a 6 with a biased die is p. It is known that p is equal to one or other of the numbers A and B where $0 < A < B < 1$. Accordingly the following statistical test of the hypothesis $H_0 : p = B$ against the alternative hypothesis $H_1 : p = A$ is performed.

The die is thrown repeatedly until a 6 is obtained. Then if X is the total number of throws, H_0 is accepted if $X \leqslant M$, where M is a given positive integer; otherwise H_1 is accepted. Let α be the probability that H_1 is accepted if H_0 is true, and let β be the probability that H_0 is accepted if H_1 is true.

Show that $\beta = 1 - \alpha^K$, where K is independent of M and is to be determined in terms of A and B. Sketch the graph of β against α.